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Incomplete Measurements-Based Finite Stabilization of Neutral Systems by Controllers with Lumped Commensurate Delays

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Abstract—This paper considers a linear autonomous differential–difference system of neutral type with lumped delays. For such systems, an output-feedback controller is proposed that simultaneously solves the finite stabilization (complete damping) problem and ensures a finite (albeit, nonarbitrary) spectrum of the closed loop system. For this controller, an existence criterion is derived and a constructive design method is presented. The distinctive feature of the controller is the absence of any distributed delay in the structure, which is important for its practical implementation. The results are illustrated by a numerical example.

Keywords: differential–difference system, neutral type, delay, finite stabilization, controller

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1. INTRODUCTION

Systems of differential equations with delay are used to model many processes in ecology, medicine, electrodynamics, deformed solid mechanics, engineering, economics, and other fields [1–3]. On the one hand, considering delay in a model improves reliability in describing real phenomena and predicting the behavior of the corresponding systems. On the other hand, incorporating process characteristics at previous time instants into the evolution law of the system increases its complexity. In this connection, quite a lot of research works have been devoted to the general theory of delayed systems and their applications (for example, see the Introduction in [3]). This paper addresses the issue of the finite stabilization of linear neutral systems with lumped delays in the state and control variables.

Stabilization problems for delayed systems are rather difficult [4–11] and have not been fully investigated to date. One possible approach is based on calculating the unstable eigenvalues of the spectrum and then replacing them with suitable numbers. However, finding such values is a nontrivial task. Therefore, a more universal method is to assign a finite spectrum to a closed loop system [12–15], usually consisting of numbers with negative real parts.

Generally speaking, the set of eigenvalues of a linear system with aftereffect is infinite, so it seems natural to control all eigenvalues of such a system by tuning the coefficients of its characteristic quasipolynomial (the problem of modal control [16–19]). Another line of stabilization-related research consists [14, 20–22] in designing a feedback controller that ensures, after a finite time, zero values for all components of the state vector of the the original open-loop system (the finite stabilization problem [23, 24], in other words, providing the complete 0-controllability by a feedback controller). An original idea for solving the finite stabilization problem is to introduce a feedback loop so that the closed loop system becomes a system with a finite spectrum pointwise degenerate in the directions corresponding to the solution components of the original system. Such ideas were further developed to systems of neutral type [15, 17, 21, 22]; a systematic presentation of these results can be found in the monograph [25].

In this paper, a finite stabilization output-feedback controller is designed for linear autonomous systems of neutral type with lumped commensurate delays. This is an output-feedback controller based on measurements of an observed signal that ensures both finite stabilization and a finite spectrum. In the case of a delayed system with scalar input and output, such a problem with the choice of any finite spectrum was studied in [24] and, for multi-input neutral systems, in [26]. A disadvantage of the approach described in [26] is the presence of distributed delay terms in the controller, although the original plant has only a lumped delay. During practical implementation, the integrals containing a distributed delay are replaced by finite sums, which may lead to undesirable consequences even when using high-precision quadrature formulas (e.g., the loss of stability) [27, 28]. The fundamental difference between this paper and [26] is the new structure of the controller, which contains purely lumped commensurate delays. The idea is to construct a discontinuous feedback defined by two controller loops (inner and outer). The inner loop "smoothens" the solution over time by using a feedback law that transforms the original system into a delayed one. After the solution reaches the necessary smoothness, the second loop is activated to ensure the pointwise degeneracy of the closed loop system in the directions corresponding to all solution components of the original (open-loop) system.

2. PROBLEM STATEMENT

Let the plant under consideration be described by a linear autonomous differential–difference system of neutral type with lumped commensurate delays:

$$\dot{x}(t) - \sum_{i=1}^{m} D_i \dot{x}(t-ih) = \sum_{i=0}^{m} \Big(A_i x(t-ih) + B_i u(t-ih) \Big), \quad t > 0,$$

$$y(t) = \sum_{i=0}^{m} C_i x(t-ih), \quad t \geqslant 0,$$

where x is the state vector of this system, u is the control input, y is the observed output, and $h = \text{const} > 0$; $D_i \in \mathbb{R}^{n\times n}$, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times r}$, and $C_i \in \mathbb{R}^{l\times n}$.

We introduce the following notations: $I_i \in \mathbb{R}^{i\times i}$ is an identity matrix, and λ_h is the shift operator defined by the rule $(\lambda_h)^k f(t) = f(t-kh)$, $k \in \mathbb{N}$, for a given value $h > 0$ and an arbitrary function f. With the polynomial matrices

$$D(\lambda) = \sum_{i=1}^{m} D_i \lambda^i, \quad A(\lambda) = \sum_{i=0}^{m} A_i \lambda^i, \quad C(\lambda) = \sum_{i=0}^{m} C_i \lambda^i, \quad B(\lambda) = \sum_{i=0}^{m} B_i \lambda^i,$$

the original plant can be written in the operator form

$$(I_n - D(\lambda_h))\dot{x}(t) = A(\lambda_h)x(t) + B(\lambda_h)u(t), \quad t > 0, \tag{1}$$

$$y(t) = C(\lambda_h)x(t), \quad t \geqslant 0. \tag{2}$$

The solution of equation (1) is uniquely determined by the initial condition

$$x(t) = \varphi(t), \quad u(t) \equiv 0, \quad t \in [-mh, 0]. \tag{3}$$

Suppose that $\varphi \in \widetilde{\mathcal{C}}^1([-mh,0],\mathbb{R}^n)$ is an unknown function, where $\widetilde{\mathcal{C}}^k(\cdot)$ indicates the class of $k-1$ times continuously differentiable functions with a piecewise continuous derivative of order k. The control input u is a piecewise continuous function.

Let $\mathbb{R}^{n\times m}[p,\lambda]$ ($\mathbb{R}^{n\times m}[\lambda]$) be the set of all matrices of dimensions $n\times m$ whose elements represent polynomials of the variables p, λ (λ) (if $m=n=1$, the superscript will be omitted), where $p_D = d/dt$ is the differentiation operator.

We define an output-feedback controller of the form

$$\begin{aligned} u(t) &= U_{11}(p_D,\lambda_h)y(t) + U_{12}(p_D,\lambda_h)\tilde{x}(t), \\ \dot{\tilde{x}}(t) &= U_{21}(p_D,\lambda_h)y(t) + U_{22}(p_D,\lambda_h)\tilde{x}(t), \quad t > t_0. \end{aligned} \tag{4}$$

Here, $\tilde{x}\in\mathbb{R}^{\tilde{n}}$ is an auxiliary variable, $t_0>0$ is some number chosen below ($u(t)\equiv 0$, $t\leqslant t_0$), $U_{11}(p,\lambda)\in\mathbb{R}^{r\times l}[p,\lambda]$, $U_{12}(p,\lambda)\in\mathbb{R}^{r\times\tilde{n}}[p,\lambda]$, $U_{21}(p,\lambda)\in\mathbb{R}^{\tilde{n}\times l}[p,\lambda]$, and $U_{22}(p,\lambda)\in\mathbb{R}^{\tilde{n}\times\tilde{n}}[p,\lambda]$. For implementing the controller (4), we specify the initial condition

$$\tilde{x}(t) = \tilde{\varphi}(t), \quad t\in[t_0-\tilde{h},t_0] \ \left(\tilde{h}=\tilde{\alpha}h, \ \ \tilde{\alpha}=\max\{\deg_\lambda U_{k2}(p,\lambda),\ k=1,2\}\right), \tag{5}$$

where $\tilde{\varphi}\in\widetilde{\mathcal{C}}^{\tilde{p}}([t_0-\tilde{h},t_0],\mathbb{R}^{\tilde{n}})$ is any function, $\tilde{p}=\max\{\deg_p U_{k2}(p,\lambda), k=1,2\}$, and the notation $\deg_\lambda f(\lambda)$ means the degree of a polynomial (including a matrix one).

The goal of this paper is to design the controller (4) ensuring the following conditions:

(a) Regardless of the initial functions φ in (3) and $\tilde{\varphi}$ in (5), there exists a number $t_1>0$ such that the vector component x of the solution vector $\mathrm{col}[x,\tilde{x}]$ of the closed loop system (1), (4) is zero starting from a time instant t_1, i.e.,

$$x(t)\equiv 0, \quad t\geqslant t_1. \tag{6}$$

(b) The closed loop system (1), (4) is a linear autonomous system of neutral type with a finite spectrum.

Remark 1. (a) By a linear autonomous homogeneous neutral system with commensurate delays we mean a linear autonomous system $\Upsilon(p_D,\lambda_h)x(t)=0$, $\Upsilon(p,\lambda)\in\mathbb{R}^{n\times n}[p,\lambda]$ with a characteristic quasipolynomial of the form $|\Upsilon(p,\lambda)|=\sum_{i=0}^{\nu}p^i\tilde{d}_i(\lambda)$, where $\nu=n\deg_p\Upsilon(p,\lambda)$, $\tilde{d}_i(\lambda)$ are polynomials, $\tilde{d}_\nu(0)=1$, and the symbol $|\cdot|$ stands for the determinant of a matrix. By introducing auxiliary variables, such a system can be rewritten as (1). Linear autonomous differential–difference systems with delay ($\tilde{d}_\nu(\lambda)\equiv 1$) and ordinary systems are treated as a special case of neutral systems.

(b) Since $U_{ij}(p,\lambda)$ are polynomial matrices, system (1), (4) has only lumped commensurate delays.

Definition 1. A controller (4) implementing conditions (a) and (b) will be called a finite stabilization output-feedback controller.

Let us denote $W(p,\lambda)=p(I_n-D(\lambda))-A(\lambda)$.

Lemma 1. *Assume that for system* (1), (2), *there exists a finite stabilization output-feedback controller* (4). *Then*

$$\operatorname{rank}\left[W(p,e^{-ph}),B(e^{-ph})\right]=n \quad \forall p\in\mathbb{C}, \tag{7}$$

$$\operatorname{rank}\left[I_n-D(\lambda),B(\lambda)\right]=n \quad \forall\lambda\in\mathbb{C}, \tag{8}$$

$$\operatorname{rank}\begin{bmatrix} W(p,e^{-ph}) \\ C(e^{-ph}) \end{bmatrix}=n \quad \forall p\in\mathbb{C}, \tag{9}$$

$$\operatorname{rank}\begin{bmatrix} I_n-D(\lambda) \\ C(\lambda) \end{bmatrix}=n \quad \forall\lambda\in\mathbb{C}. \tag{10}$$

The proof is postponed to the Appendix.

3. THE MAIN RESULT

Now we formulate the main result of this paper.

Theorem 1. *For system* (1), (2) *there exists a finite stabilization output-feedback controller* (4) *iff conditions* (7)–(10) *are valid.*

Proof. Necessity follows from Lemma 1. **Sufficiency.** The sufficient nature of the conditions of Theorem 1 will be established in two parts. In the first part, we design a controller implementable under the condition that the output $y(t)$ is a $(\rho_0 - 1)$ times continuously differentiable function with a piecewise continuous derivative of order ρ_0, where the number ρ_0 is determined when constructing the controller (see Remark 2). To satisfy the above condition for the function $y(t)$, we suppose that $\varphi \in \widetilde{\mathcal{C}}^{\rho_0}$. The second part of the proof considers the general case $\varphi \in \widetilde{\mathcal{C}}^1$ and $\rho_0 > 1$, i.e., the smoothness of the initial function does not ensure the same property for the output $y(t)$, which is described above.

3.1. The Case $\varphi \in \widetilde{\mathcal{C}}^{\rho_0}$

To prove the sufficiency of the theorem's conditions, we design the controller (4). The design process will consist of the following steps: 1) constructing a finite stabilization state-feedback controller; 2) constructing a finite observer; 3) designing a finite stabilization output-feedback controller based on the parameters of the controller and observer constructed at the previous steps.

1. Constructing a finite stabilization state-feedback controller

Due to (7) and (8), for system (1), there exists [22; 25, p. 358] a controller (further called a finite stabilization state-feedback controller) of the form

$$\begin{aligned} u(t) &= L_{00}(p_D, \lambda_h)x(t) + L_{01}(p_D, \lambda_h)\bar{x}(t), \\ \dot{\bar{x}}(t) &= L_{10}(p_D, \lambda_h)x(t) + L_{11}(p_D, \lambda_h)\bar{x}(t), \quad t > 0, \end{aligned} \tag{11}$$

where $\bar{x} \in \mathbb{R}^{\bar{n}}$ is an auxiliary variable, $L_{00}(p,\lambda) \in \mathbb{R}^{r\times n}[p,\lambda]$, $L_{01}(p,\lambda) \in \mathbb{R}^{r\times \bar{n}}[p,\lambda]$, $L_{10}(p,\lambda) \in \mathbb{R}^{\bar{n}\times n}[p,\lambda]$, $L_{11}(p,\lambda) \in \mathbb{R}^{\bar{n}\times\bar{n}}[p,\lambda]$, and $\deg_p L_{ij}(p,\lambda) = 1$, with the following conditions:

(1) It is possible to find a number $\bar{t}_1 > 0$ such that, regardless of the initial condition of system (1), (11), we have

$$x(t) \equiv 0, \quad t \geqslant \bar{t}_1. \tag{12}$$

(2) System (1), (11) is a linear autonomous neutral system with lumped commensurate delays and a finite (albeit, not a priori given) spectrum. Since the spectrum of the closed loop system is finite, the determinant of the characteristic matrix of this system will be a polynomial, i.e.,

$$|W_0(p,\lambda)| = d_0(p). \tag{13}$$

Here, $d_0(p)$ is some polynomial and $W_0(p, e^{-ph})$ is the characteristic matrix of system (1), (11) given by

$$W_0(p,\lambda) = \begin{bmatrix} W(p,\lambda) - B(\lambda)L_{00}(p,\lambda) & -B(\lambda)L_{01}(p,\lambda) \\ -L_{10}(p,\lambda) & pI_{\bar{n}} - L_{11}(p,\lambda) \end{bmatrix}. \tag{14}$$

We present the idea of constructing the controller (11) [22; 25, p. 358]. Conditions (7) and (8) are necessary and sufficient for the existence of matrices $L_{ij}(p,\lambda)$ in (11) such that the system corresponding to the matrix (14) is pointwise degenerate in the directions $\bar{e}_i$, $i = \overline{1, n+\bar{n}-1}$, where $\bar{e}_i$ is the ith column of the matrix $I_{n+\bar{n}}$. This implies [29] the existence of a time instant $\bar{t}_1$ such that $\bar{e}_i'\mathrm{col}[x(t), \bar{x}(t)] \equiv 0$, $t \geqslant \bar{t}_1$, $i = \overline{1, n+\bar{n}-1}$. (The prime $'$ indicates transpose.) The latter identity ensures (12). The construction procedure for the matrices $L_{ij}(p, \lambda)$ from (11) was described in [22; 25, p. 358].

2. Constructing a finite observer

By a finite observer we mean [30, 31] a linear autonomous delayed differential system dependent on the output (3) with lumped commensurate delays, a finite spectrum, and the output v that has the following property: there exists a time instant $t_* > 0$ starting from which, regardless of the initial conditions of the observer and equation (1), the observer's output v is equal to the solution x of equation (1) generating the output y, i.e., $x(t) = v(t)$, $t \geqslant t_*$.

As was shown in [30, 31], conditions (9) and (10) are necessary and sufficient for the existence of a finite observer. In this case, the observer can be constructed both as a system with distributed delays and any given finite spectrum [30] and as a system without distributed delays with a finite (albeit, not a priori given) spectrum [31]. For the goal of this paper, we will modify one observer from [31].

By condition (10), there are matrices $L_1(\lambda) \in \mathbb{R}^{n\times l}[\lambda]$ and $L_2(\lambda) \in \mathbb{R}^{l\times l}[\lambda]$ such that [17, 22]

$$|I_{n+l} - D_L(\lambda)| \equiv 1, \quad D_L(\lambda) = \begin{bmatrix} D(\lambda) & \lambda L_1(\lambda) \\ C(\lambda) & \lambda L_2(\lambda) \end{bmatrix}. \tag{15}$$

Let $\Pi(\lambda) = [\Pi_{ij}(\lambda)]_{i,j=1}^2$ be the adjoint matrix for the matrix $(I_{n+l} - D_L(\lambda))$, where $\Pi_{11}(\lambda) \in \mathbb{R}^{n\times n}[\lambda]$, $\Pi_{12}(\lambda) \in \mathbb{R}^{n\times l}[\lambda]$, $\Pi_{21}(\lambda) \in \mathbb{R}^{l\times n}[\lambda]$, and $\Pi_{22}(\lambda) \in \mathbb{R}^{l\times l}[\lambda]$. From (15) it follows that $\Pi(\lambda) = (I_{n+l} - D_L(\lambda))^{-1}$. We introduce the new function

$$\chi(t) = (I_n - D(\lambda_h))x(t), \quad t \geqslant 0. \tag{16}$$

Let $\tilde{\chi}(t)$ ($\tilde{\chi} \in \mathbb{R}^l$, $t \in \mathbb{R}$) be an arbitrary function. Applying the operator $\Pi(\lambda_h)$ to the equality

$$\begin{bmatrix} I_n - D(\lambda_h) & -\lambda_h L_1(\lambda_h) \\ -C(\lambda_h) & I_l - \lambda_h L_2(\lambda_h) \end{bmatrix} \begin{bmatrix} x(t) \\ \tilde{\chi}(t) \end{bmatrix} = \begin{bmatrix} \chi(t) \\ -y(t) \end{bmatrix} + \begin{bmatrix} -\lambda_h L_1(\lambda_h)\tilde{\chi}(t) \\ (I_l - \lambda_h L_2(\lambda_h))\tilde{\chi}(t) \end{bmatrix}, \quad t \geqslant 0,$$

on the left allows establishing the relation

$$x(t) = \Pi_{11}(\lambda_h)\chi(t) - \Pi_{12}(\lambda_h)y(t), \quad t \geqslant \gamma_2 h, \tag{17}$$

where $\gamma_2 = \max\{\nu_{1j},\ j = 1, 2\}$ and $\nu_{ij} = \deg_\lambda \Pi_{ij}(\lambda)$. Next, let us denote

$$\widetilde{A}(\lambda) = A(\lambda)\Pi_{11}(\lambda), \quad \widetilde{C}(\lambda) = \begin{bmatrix} C(\lambda)\Pi_{11}(\lambda) \\ (I_n - D(\lambda))\Pi_{11}(\lambda) - I_n \end{bmatrix},$$

$$\tilde{y}(t) = C_y(\lambda_h)y(t), \quad t \geqslant \gamma_3 h, \quad C_y(\lambda) = \begin{bmatrix} I_l + C(\lambda)\Pi_{12}(\lambda) \\ (I_n - D(\lambda))\Pi_{12}(\lambda) \end{bmatrix}, \quad \gamma_3 = m + \gamma_2.$$

Based on (16) and (17), system (1), (2) can be written as an inhomogeneous linear autonomous differential–difference system with commensurate delays and the known output $\tilde{y}$:

$$\dot{\chi}(t) = \widetilde{A}(\lambda_h)\chi(t) + B(\lambda_h)u(t) - A(\lambda_h)\Pi_{12}(\lambda_h)y(t), \quad t > \gamma_3 h, \tag{18}$$

$$\tilde{y}(t) = \widetilde{C}(\lambda_h)\chi(t), \quad t \geqslant \gamma_3 h. \tag{19}$$

In view of (9), system (18), (19) satisfies the condition [30, 31]

$$\operatorname{rank}\begin{bmatrix} pI_n - \widetilde{A}(e^{-ph}) \\ \widetilde{C}(e^{-ph}) \end{bmatrix} = n \quad \forall p \in \mathbb{C}. \tag{20}$$

From (20) it follows that, for any $i_0 \in \{1, \ldots, n+l\}$, there exists a matrix $V_{i_0}(\lambda) \in \mathbb{R}^{n\times(n+l)}[\lambda]$ such that

$$\operatorname{rank}\begin{bmatrix} pI_n - \widetilde{A}(e^{-ph}) - V_{i_0}(e^{-ph})\widetilde{C}(e^{-ph}) \\ \tilde{c}_{i_0}(e^{-ph}) \end{bmatrix} = n \quad \forall p \in \mathbb{C}, \tag{21}$$

where $\tilde{c}_{i_0}(\lambda)$ is the i_0th row of the matrix $\widetilde{C}(\lambda)$ [12]. Letting

$$\widetilde{A}_V(\lambda) = \widetilde{A}(\lambda) + V_{i_0}(\lambda)\widetilde{C}(\lambda), \quad K_0(\lambda) = -A(\lambda)\Pi_{12}(\lambda_h) - V_{i_0}(\lambda)C_y(\lambda) \tag{22}$$

and using equations (18), (19) and formulas (22), we replace system (1), (2) with

$$\begin{aligned} \dot{\chi}(t) &= \widetilde{A}_V(\lambda_h)\chi(t) + B(\lambda_h)u(t) + K_0(\lambda_h)y(t), \quad t > \tilde{t}_1, \\ \tilde{y}_{i_0}(t) &= \tilde{c}_{i_0}(\lambda_h)\chi(t), \quad t \geqslant \tilde{t}_1, \end{aligned} \tag{23}$$

where $\tilde{y}_{i_0}(t)$ is the i_0th component of the vector $\tilde{y}$, $\tilde{t}_1 = (\nu_0 + \gamma_3)h$, and $\nu_0 = \deg_\lambda V_{i_0}(\lambda)$.

Due to condition (21), for system (23), there exists [31] a finite observer in the form of a finite-spectrum system with purely lumped commensurate delays:

$$\dot{z}(t) = Q(p_D, \lambda_h)z(t) + K(\lambda_h)y(t) + \overline{B}(\lambda_h)u(t), \quad t > \tilde{t}_1; \tag{24}$$

in addition, the output v_z determining the estimate of the solution χ of system (23) is given by

$$v_z(t) = [I_n, 0_{n\times 3}]z(t), \quad t \geqslant \tilde{t}_1. \tag{25}$$

Here, $z = \operatorname{col}[z_1, z_2]$, $z_1 \in \mathbb{R}^n$, $z_2 \in \mathbb{R}^3$, $z_1 = \operatorname{col}[z_{11}, \ldots, z_{1n}]$, $z_2 = \operatorname{col}[z_{21}, z_{22}, z_{23}]$, $Q(p, \lambda) \in \mathbb{R}^{(n+3)\times(n+3)}[p, z]$, $0_{n\times m}$ denotes a zero matrix of dimensions $n \times m$,

$$\overline{B}(\lambda) = \begin{bmatrix} B(\lambda) \\ 0_{3\times r} \end{bmatrix}, \tag{26}$$

and the matrix $K(\lambda)$ is found from the equality

$$K(\lambda_h)y(t) = \begin{bmatrix} K_0(\lambda_h) \\ 0_{3\times l} \end{bmatrix} y(t) - e_{n+1}\tilde{y}_{i_0}(t) = \left(\begin{bmatrix} K_0(\lambda_h) \\ 0_{3\times l} \end{bmatrix} - e_{n+1}\,\tilde{e}'_{i_0}C_y(\lambda_h)\right) y(t), \tag{27}$$

where e_i and $\tilde{e}_i$ are the ith columns of the matrices I_{n+3} and I_{n+l}, respectively. The matrix $Q(p, \lambda)$ is obtained by the scheme for constructing the finite observer matrix for a homogeneous delayed system with scalar output [31]. The elements of the matrix $Q(p, \lambda)$ are such that, after introducing auxiliary variables, the homogeneous system (24) can be written in the standard form of a linear autonomous delayed system (i.e., as $\dot{X}(t) = \Sigma(\lambda_h)X(t)$, where $\Sigma(\lambda)$ is a polynomial matrix), and

$$|pI_{n+3} - Q(p, \lambda)| = d_1(p), \tag{28}$$

where $d_1(\lambda)$ is a polynomial. The matrix $Q(p,\lambda)$ has the form

$$Q(p,\lambda)=\left[\begin{array}{ccc|ccc} \tilde a^V_{11}(\lambda) & \dots & \tilde a^V_{1n}(\lambda) & g_{11}(\lambda) & \tilde g_{12} & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \tilde a^V_{n1}(\lambda) & \dots & \tilde a^V_{nn}(\lambda) & g_{n1}(\lambda) & \tilde g_{n2} & 0\\ \hline \tilde c^1_{i_0}(\lambda) & \dots & \tilde c^n_{i_0}(\lambda) & g_{n+1\,1}(p,\lambda) & 1 & 0\\ 0 & \dots & 0 & \lambda g_{n+2\,1}(p,\lambda) & g_{n+2\,2}(p,\lambda) & g_{n+2\,3}(\lambda)\\ 0 & \dots & 0 & \lambda g_{n+3\,1}(\lambda) & g_{n+3\,2}(\lambda) & g_{n+3\,3}(\lambda) \end{array}\right], \tag{29}$$

where $\tilde a^V_{ij}(\lambda)$ are the elements of the matrix $\widetilde A_V(\lambda)$, $\widetilde A_V(\lambda)=[\tilde a^V_{ij}(\lambda)]_{n\times n}$, $\tilde c^j_{i_0}(\lambda)$ are the elements of the vector $\tilde c_{i_0}(\lambda)$, $\tilde c_{i_0}(\lambda)=[\tilde c^1_{i_0}(\lambda),\dots,\tilde c^n_{i_0}(\lambda)]$, $g_{ij}(p,\lambda)$ and $g_{ij}(\lambda)$ are polynomials of the variables p, λ and λ, respectively, and $\tilde g_{i2}\in\mathbb{R}$.

Remark 2. Let $\rho_0=\max\{\deg_p g_{n+2\,1}(p,\lambda)-\deg_p(g_{n+1\,1}(p,\lambda)-p),1\}$. The component z_{21} depends on the output y; therefore, $z_{21}\in\widetilde{\mathcal C}^{\rho_0}([\tilde t_1,+\infty),\mathbb R)$ is a necessary condition for the term $\lambda_h g_{n+2\,1}(p_D,\lambda_h)z_{21}$ to exist in system (24). Hence, we require $\tilde y_{i_0}\in\widetilde{\mathcal C}^{\rho_0}([\tilde t_1,+\infty),\mathbb R)$, which is achieved by $\varphi\in\widetilde{\mathcal C}^{\rho_0}([-mh,0],\mathbb R^n)$.

The components of the initial function $z(t)$, $t\in[\tilde t_1-h_0,\tilde t_1]$ (h_0 specifies the delay of system (24)), are taken smooth enough with a piecewise continuous senior derivative. (For each component, the order of this derivative is determined by the maximum degree of the variable p of the corresponding polynomials in the matrix (29).) In particular, it is possible to set $z(t)\equiv 0$, $t\in[t_0-h_0,t_0]$.

Now we explain the idea of choosing the elements of the matrix $Q(p,\lambda)$. Let $\zeta=v_z-\chi=z_1-\chi$ denote the estimation error and $\tilde\zeta=\mathrm{col}\,[\zeta,z_2]$. In view of (29) and (25), the vector function $\tilde\zeta(t)$ is given by the linear autonomous delayed system

$$\dot{\tilde\zeta}(t)=Q(p_D,\lambda_h)\tilde\zeta(t),\quad t>\tilde t_1. \tag{30}$$

The elements of the matrix $Q(p,\lambda)$ are chosen so that system (30) is pointwise degenerate in the directions corresponding to the first $(n+2)$ columns of the matrix I_{n+3}, i.e., in the directions e_i, $i=\overline{1,n+2}$. Hence, there exists a time instant $\tilde t_2$ such that $e_i'\tilde\zeta(t)\equiv 0$, $t\geqslant\tilde t_2$, $i=\overline{1,n+2}$, regardless of the initial function defining the solution of system (30). Consequently, the equality

$$\chi(t)=v_z(t),\quad t\geqslant\tilde t_2, \tag{31}$$

holds for any initial functions of systems (1) and (24).

Finally, we estimate the solution of system (1), (2) using formula (17). With

$$v(t)=\Pi_{11}(\lambda_h)[I_n,0_{n\times 3}]z(t)-\Pi_{12}(\lambda_h)y(t),\quad t\geqslant\tilde t_1, \tag{32}$$

from equality (31) and formula (17) it follows that

$$x(t)=v(t),\quad t\geqslant\tilde t_3, \tag{33}$$

where $\tilde t_3=\tilde t_2+\nu_{11}h$. Thus, the finite observer (24), (32) has been constructed.

3. Designing a finite stabilization output-feedback controller

Let us derive expressions for the controller (4). To this end, the control inputs $u(t)$ in equations (24) are replaced using the first formula of (11); the variable x in the resulting equation and the relations (11) is expressed through z, y using (33) and (32). Next, denoting the variables $\bar{x}$, z by x_1, x_2, respectively, we write the controller

$$u(t) = R_{01}(p_D, \lambda_h)x_1(t) + R_{02}(p_D, \lambda_h)x_2(t) + R_{00}(p_D, \lambda_h)y(t), \tag{34}$$

$$\dot{x}_1(t) = R_{11}(p_D, \lambda_h)x_1(t) + R_{12}(p_D, \lambda_h)x_2(t) + R_{10}(p_D, \lambda_h)y(t), \tag{35}$$

$$\begin{aligned} \dot{x}_2(t) = R_{22}(p_D, \lambda_h)x_2(t) + \overline{B}(\lambda_h)\Big(R_{01}(p_D, \lambda_h)x_1(t) \\ + R_{02}(p_D, \lambda_h)x_2(t) + R_{00}(p_D, \lambda_h)y(t)\Big) + K(\lambda_h)y(t), \quad t > t_0, \end{aligned} \tag{36}$$

where $x_i \in \mathbb{R}^{n_i}$, $i = 1, 2$ $(n_1 = \bar{n},\ \ n_2 = n + 3)$, are auxiliary variables, $t_0 = \alpha_0 h$, $\alpha_0 = \max\{\deg_\lambda R_{00}(p, \lambda) + m, \deg_\lambda R_{10}(p, \lambda), \deg_\lambda K(\lambda)\}$, and

$$\begin{aligned} R_{i0}(p, \lambda) = -L_{i0}(p, \lambda)\Pi_{12}(\lambda), \quad R_{i1}(p, \lambda) = L_{i1}(p, \lambda), \\ R_{i2}(p, \lambda) = L_{i0}(p, \lambda)\Pi_{11}(\lambda)[I_n, 0_{n\times 3}], \quad i = 0, 1, \quad R_{22}(p, \lambda) = Q(p, \lambda). \end{aligned} \tag{37}$$

Letting $\tilde{x} = \mathrm{col}[x_1, x_2]$, $U_{11}(p, \lambda) = R_{00}(p, \lambda)$, $U_{12}(p, \lambda) = \mathrm{col}[R_{01}(p, \lambda), R_{02}(p, \lambda)]$, and

$$U_{21}(p, \lambda) = \begin{bmatrix} R_{10}(p, \lambda) \\ \overline{B}(\lambda)R_{00}(p, \lambda) + K(\lambda) \end{bmatrix},$$

$$U_{22}(p, \lambda) = \begin{bmatrix} R_{11}(p, \lambda) & R_{12}(p, \lambda) \\ \overline{B}(\lambda)R_{01}(p, \lambda) & R_{22}(p, \lambda) + \overline{B}(\lambda)R_{02}(p, \lambda) \end{bmatrix}$$

allows representing the controller (34)–(36) in the form (4).

Let $\hat{e}_i$ be the columns of the identity matrix $I_{n+n_1+n_2}$.

Proposition 1. *System (1), (2), (34)–(36) is pointwise degenerate in the directions $\hat{e}_i$, $i = \overline{1, n + n_1 - 1}$, $i = \overline{n + n_1 + 1, n + n_1 + n_2 - 1}$, and the set of its spectral values and their multiplicity are determined by the roots of the polynomial $d_0(\lambda)d_1(\lambda)$.*

The proof is provided in the Appendix.

By Proposition 1, the constructed controller (34)–(36) is a finite stabilization output-feedback controller. In the case $\varphi \in \widetilde{\mathcal{C}}^{\rho_0}$, Theorem 1 is proved.

3.2. The Case $\varphi \in \widetilde{\mathcal{C}}^1$

If $\rho_0 = 1$ (see Remark 2), then the controller (34)–(36) is the desired finite stabilization controller and the considerations of Section 3.2 become unnecessary. In what follows, we assume that $\rho_0 > 1$.

The finite stabilization output-feedback controller will be constructed as a variable structure (discontinuous feedback) controller [33] consisting of two serially connected loops: inner $\widehat{u}$ and outer v:

$$u(t) = \begin{cases} 0, & t \leqslant t_5 \\ \widehat{u}(t), & t \in (t_5, t_6] \\ \widehat{u}(t) + v(t), & t > t_6. \end{cases} \tag{38}$$

The inner loop $\widehat{u}$ ensures "smoothing" of the solution of the corresponding closed-loop system (1) over time. Once the solution of the system is $\rho_0 - 1$ times continuously differentiable and has a piecewise continuous derivative of order ρ_0, the outer loop v (34)–(36) is activated to ensure the pointwise degeneracy of the closed loop system.

Remark 3. In general, the loops $\hat{u}$ and v may contain auxiliary variables as their arguments. Therefore, the full description of the finite stabilization output-feedback controller will be the relation (38) supplemented by differential equations with initial conditions describing the behavior of the auxiliary variables similar to the relations (4) and (5).

Let us impose a condition on the parameters of the homogeneous ($u \equiv 0$) system (1) under which the smoothness of its solution will increase over time. We denote by $\Pi_D(\lambda)$ the adjoint matrix for the matrix $(I_n - D(\lambda))$, $m_0 = \deg_\lambda A(\lambda)\Pi_D(\lambda)$.

Lemma 2. *Assume that the homogeneous* ($u \equiv 0$) *system* (1) *satisfies the condition*

$$\big|I_n - D(\lambda)\big| \equiv 1 \tag{39}$$

and $\varphi \in \widetilde{\mathcal{C}}^1$ *in the initial condition* (3)*. Then, for any* $\rho_1 \in \mathbb{N}$ *and the solution* x *of system* (1)*, we have* $x \in \widetilde{C}^{\rho_1}\big([t_4 + \rho_1 m_0 h, +\infty), \mathbb{R}^n\big)$*, where* $t_4 = h \deg_\lambda \Pi_D(\lambda)$*.*

The proof is given in the Appendix.

Remark 4. The identity (39) is equivalent to the fact that the characteristic quasipolynomial of system (1) has the form $\big|W(p,\lambda)\big| = p^n + \sum_{i=0}^{n-1} p^i \hat{d}_i(\lambda)$, where $\hat{d}_i(\lambda)$ are polynomials.

Remark 5. By the proof of Lemma 2 (see the Appendix), under (39), the homogeneous system of neutral type is reduced, through a nondegenerate change of the variables, to a delayed system whose solution will smoothen over time. Let us present other considerations showing that if (39) holds for the homogeneous system of neutral type, the smoothness of the solution will increase with t. For clarity, let $D_1 \neq 0$ and $D_i = 0$, $i = \overline{2,m}$, i.e., system (1) has the form

$$\dot{x}(t) - D_1\dot{x}(t-h) = A(\lambda_h)x(t), \quad t > 0.$$

Then we obtain the following chain of equalities:

$$\begin{aligned} \dot{x}(t) &= A(\lambda_h)x(t) + D_1\dot{x}(t-h) \\ &= A(\lambda_h)x(t) + D_1\big(A(\lambda_h)x(t-h) + D_1\dot{x}(t-2h)\big) \\ &= \ldots = \sum_{i=0}^{\tilde{m}-1} D_1^i A(\lambda_h)x(t-ih) + D^{\tilde{m}}\dot{x}(t-\tilde{m}h), \quad t > \tilde{m}h, \quad \tilde{m} \in \mathbb{N}. \end{aligned} \tag{40}$$

In this case ($D_1 \neq 0$ and $D_i = 0$, $i = \overline{2,m}$), condition (39) becomes $\big|I_n - \lambda D_1\big| \equiv 1$. This means that the matrix D_1 is nilpotent. Let $\tilde{m}_0$ be the nilpotency index of the matrix D_1, $D_1^{\tilde{m}_0} = 0$. Then from (40) we obtain

$$\dot{x}(t) = \sum_{i=0}^{\tilde{m}_0-1} D_1^i A(\lambda_h)x(t-ih), \quad t > \tilde{m}_0 h. \tag{41}$$

System (41) is a delayed system with $m(\tilde{m}_0 - 1)$ commensurate delays. Therefore, for $t > k\tilde{m}_0 h$, $k = 1, 2, \ldots$, the smoothness of the solution increases by k units.

Similar reasoning is valid for an arbitrary polynomial matrix $\widetilde{D}(\lambda)$. (Condition (39) as necessary and sufficient for the nilpotency of some matrix at the derivatives of the solution containing delays was discussed in [25, p. 218]; see Lemma 4.10.)

Lemma 3. *Under conditions* (8) *and* (10)*, there exist matrices* $\widetilde{U}_{11}(\lambda) \in \mathbb{R}^{r\times l}[\lambda]$*,* $\widetilde{U}_{12}(\lambda) \in \mathbb{R}^{r\times(r+n+l)}[\lambda]$*,* $\widetilde{U}_{21}(\lambda) \in \mathbb{R}^{(n+r+l)\times n}[\lambda]$*, and* $\widetilde{U}_{22}(\lambda) \in \mathbb{R}^{(r+n+l)\times(r+n+l)}[\lambda]$ *such that*

$$\begin{gathered} \big|I_{2n+r+l} - \widetilde{D}(\lambda)\big| \equiv 1, \\ \widetilde{D}(\lambda) = \begin{bmatrix} D(\lambda) + B(\lambda)\widetilde{U}_{11}(\lambda)C(\lambda) & B(\lambda)\widetilde{U}_{12}(\lambda) \\ \widetilde{U}_{21}(\lambda)C(\lambda) & \widetilde{U}_{22}(\lambda) \end{bmatrix}, \ \widetilde{D}(0) = 0_{(2n+r+l)\times(2n+r+l)}. \end{gathered} \tag{42}$$

The proof is postponed to the Appendix.

We define the inner loop controller by the relations

$$\begin{aligned} \widetilde{u}(t) &= p_{_D}\widetilde{U}_{11}(\lambda_h)y(t) + p_{_D}\widetilde{U}_{12}(\lambda_h)x_3(t) + v_1(t), \\ \dot{x}_3(t) &= p_{_D}\widetilde{U}_{21}(\lambda_h)y(t) + p_{_D}\widetilde{U}_{22}(\lambda_h)x_3(t) + v_2(t), \quad t > t_5, \end{aligned} \tag{43}$$

where $x_3 \in \mathbb{R}^{n+r+l}$ is an auxiliary variable, $v = \operatorname{col}[v_1, v_2]$, the matrices $\widetilde{U}_{ij}(\lambda)$ ensure (42), $t_5 = h \max\{m + \deg_\lambda \widetilde{U}_{11}(\lambda), \deg_\lambda \widetilde{U}_{21}(\lambda)\}$. We write system (1), (43):

$$\begin{aligned} (I_{2n+r+l} - \widetilde{D}(\lambda)) \begin{bmatrix} \dot{x}(t) \\ \dot{x}_3(t) \end{bmatrix} &= \begin{bmatrix} A(\lambda_h) & 0_{n\times(n+r+l)} \\ 0_{(n+r+l)\times n} & 0_{(n+r+l)\times(n+r+l)} \end{bmatrix} \begin{bmatrix} x(t) \\ x_3(t) \end{bmatrix} \\ &+ \begin{bmatrix} B(\lambda_h) & 0_{n\times(n+r+l)} \\ 0_{(n+r+l)\times r} & I_{n+r+l} \end{bmatrix} v(t), \quad t > t_5. \end{aligned} \tag{44}$$

Due to the condition $\widetilde{D}(0) = 0_{(2n+r+l)\times(2n+r+l)}$, system (44) has neutral type; in view of (42), it also satisfies the condition of Lemma 2.

Let us specify the initial condition $x_3(t) = \varphi_3(t)$, $t \in [t_5 - h_3, t_5]$, where $\varphi_3 \in \mathcal{C}^1([t_5 - h_3, t_5], \mathbb{R}^{n+r+l})$ is any function and $h_3 = h \max\left\{\deg_\lambda \widetilde{W}_{13}(\lambda), \deg_\lambda \widetilde{W}_{23}(\lambda)\right\}$.

For system (44) we add the output signal

$$y_1(t) = \begin{bmatrix} C(\lambda_h) & 0_{l\times(n+r+l)} \\ 0_{(n+r+l)\times n} & I_{n+r+l} \end{bmatrix} \begin{bmatrix} x(t) \\ x_3(t) \end{bmatrix}, \tag{45}$$

where $y_1(t) = \operatorname{col}[y(t), x_3(t)]$. Clearly, system (44), (45) satisfies the conditions of Theorem 1.

Let $v(t) = 0$, $t \leqslant t_6$, in system (44). For $t > t_6$, we construct the loop v according to the scheme of Section 3.1 but for system (44), (45). The number t_6 is appropriately chosen to fulfill the smoothness requirement described in Remark 2.

Remark 6. In several cases, there may exist a polynomial matrix $\widetilde{U}(\lambda)$ such that $|I_n - D(\lambda) - \lambda B(\lambda)\widetilde{U}(\lambda)C(\lambda)| \equiv 1$. Then, to reduce the size of the matrices of the finite stabilization output-feedback controller, we should take the inner loop controller in the form $\widetilde{u}(t) = p_{_D}\widetilde{U}(\lambda_h)y(t) + v(t)$ instead of (43). In this case, the output (45) is replaced by the output (3), whereas the variable x_3 and the corresponding blocks in (44) disappear (see the example below).

Example 1. We demonstrate the method of constructing a finite stabilization controller of the form (4) (see the proof of Theorem 1) on an example of system (1), (2) with $h = \ln 2$ and the matrices

$$D(\lambda) = \begin{bmatrix} \lambda + \lambda^2 & 0 \\ \lambda^2 & 0 \end{bmatrix}, \ A(\lambda) = \begin{bmatrix} 1-\lambda & 1 \\ 0 & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ C(\lambda) = \begin{bmatrix} 1+\lambda, & 0 \end{bmatrix}. \tag{46}$$

In this case, the conditions of Theorem 1 are valid. In accordance with Remark 6, we find

$$\widetilde{u}(t) = p_{_D}[\lambda_h]y(t) + v(t), \quad t > t_5 = 2h. \tag{47}$$

(Here, $[\lambda_h]$ is a matrix of dimensions 1×1.)

For the case (46), (47), system (44), (45) takes the form

$$\left(I_2 - \begin{bmatrix} 0 & 0 \\ -\lambda_h & 0 \end{bmatrix}\right)\dot{x}(t) = \begin{bmatrix} 1-\lambda_h & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} v(t), \ \ y(t) = [1+\lambda_h, 0]x(t), \ \ t > t_5. \tag{48}$$

Interpreting it as system (1), (2), we follow steps 1)–3) of Section 3.1.

1. The controller (11) is constructed as described in [22]:

$$\begin{aligned} v(t) &= \left[-\frac{2}{3}\lambda_h^3 + \lambda_h^2 + \frac{8}{3}\lambda_h - 2,\ -\frac{2}{3}\lambda_h^2 + \lambda_h - \frac{4}{3}\right] x(t) \\ &+ \left[\lambda_h^3 - \frac{7}{2}\lambda_h^2 + \frac{7}{2}\lambda_h - 1\right] \bar{x}(t), \\ \dot{\bar{x}}(t) &= \left[-\frac{4}{9}\lambda_h^2 + \frac{10}{9}\lambda_h + \frac{8}{3},\ \frac{4}{9}\lambda_h + \frac{10}{9}\right] x(t) + \left[\frac{2}{3}\lambda_h^2 - 3\lambda_h + \frac{7}{3}\right] \bar{x}(t). \end{aligned} \tag{49}$$

Consider system (48) closed with the controller (49). The characteristic matrix $W_0(p, \lambda)$ (see (14)) has the form

$$W_0(p,\lambda) = \begin{bmatrix} p + 1 + \frac{2}{3}\lambda^3 - \lambda^2 - \frac{5}{3}\lambda & \frac{2}{3}\lambda^2 - \lambda + \frac{1}{3} & -\lambda^3 + \frac{7}{2}\lambda^2 - \frac{7}{2}\lambda + 1 \\ p\lambda + \frac{2}{3}\lambda^3 - \lambda^2 - \frac{8}{3}\lambda + 2 & p + \frac{2}{3}\lambda^2 - \lambda + \frac{4}{3} & -\lambda^3 + \frac{7}{2}\lambda^2 - \frac{7}{2}\lambda + 1 \\ \frac{4}{9}\lambda^2 - \frac{10}{9}\lambda - \frac{8}{3} & \frac{4}{9}\lambda - \frac{10}{9} & p - \frac{2}{3}\lambda^2 + 3\lambda - \frac{7}{3} \end{bmatrix}. \tag{50}$$

Direct verification shows that $d_0(p) = p^3 - p$. To investigate pointwise degeneracy we can apply, e.g., [29, Theorem 1.1]. Let us briefly illustrate this process. The elements of the first two rows of the matrix adjoint to the matrix $W_0(p, e^{-ph})$ in (50) vanish on the roots of the polynomial $d_0(p)$; therefore, the elements of the first two rows of the matrix $\left(W_0(p, e^{-ph})\right)^{-1}$ are integer functions. This property implies [29] pointwise degeneracy in the directions $[1, 0, 0]$ and $[0, 1, 0]$, i.e., condition (12) holds. The maximum degree of the variable λ in these rows does not exceed 5, so $\bar{t}_1 = 5h$.

2. We construct the finite observer (24), (32). In the case under consideration,

$$D_L(\lambda) = \begin{bmatrix} \lambda & 0 & -\frac{\lambda}{2} \\ 0 & 0 & 0 \\ 1+\lambda & 0 & -\frac{\lambda}{2} \end{bmatrix}, \quad \Pi(\lambda) = \begin{bmatrix} \frac{\lambda}{2} + 1 & 0 & -\frac{\lambda}{2} \\ 0 & 1 & 0 \\ 1+\lambda & 0 & 1-\lambda \end{bmatrix},$$

$$\widetilde{C}(\lambda) = \begin{bmatrix} \frac{\lambda^2}{2} + \frac{3}{2}\lambda + 1 & 0 \\ -\frac{\lambda^2}{2} - \frac{\lambda}{2} & 0 \\ 0 & 0 \end{bmatrix}, \quad C_y(\lambda) = \begin{bmatrix} -\frac{\lambda^2}{2} + \frac{\lambda}{2} + 1 \\ \frac{\lambda^2}{2} - \frac{\lambda}{2} \\ 0 \end{bmatrix},$$

$$V_2(\lambda) = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ (i_0 = 2).$$

System (23) takes the form

$$\dot{\chi}(t) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \chi(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t), \quad \tilde{y}_2(t) = \left[-\frac{\lambda_h^2}{2} - \frac{\lambda_h}{2},\ 0\right] \chi(t). \tag{51}$$

Using (51), we finally arrive at the relations (24), (32):

$$\dot{z}(t) = Q(p_D, \lambda_h)z(t) + \begin{bmatrix} 0 \\ 0 \\ -\dfrac{\lambda_h^2}{2} + \dfrac{\lambda_h}{2} \\ 0 \\ 0 \end{bmatrix} y(t) + \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u(t),$$

$$x(t) = \begin{bmatrix} \dfrac{\lambda_h}{2} + 1 & 0 \\ 0 & 1 \end{bmatrix} z_1(t) + \begin{bmatrix} \dfrac{\lambda_h}{2} \\ 0 \end{bmatrix} y(t).$$

Here are the elements of the matrix $Q(p, \lambda)$ located in blocks nos. (1,2) and (2,2):

$$g_{11}(\lambda) = 0, \quad g_{21}(\lambda) = 0, \quad g_{31}(p, \lambda) = -1,$$

$$\begin{aligned} g_{41}(p, \lambda) = {} & \frac{428\,259\,827\,248}{370\,825\,875} + \frac{13\,308\,418}{37\,975}p + \frac{3\,263\,970\,139}{410\,130}p\lambda^2 - \frac{64\,061\,677\,864\,590\,419}{683\,506\,252\,800}\lambda^7 \\ & - \frac{4\,504\,350\,207\,517}{370\,825\,875}\lambda + \frac{10\,314\,197}{36\,325\,800}\lambda^{14} + \frac{109\,094\,554\,247\,916\,287}{683\,506\,252\,800}\lambda^6 \\ & - \frac{17\,328\,104\,121\,953\,0591}{854\,382\,816\,000}\lambda^5 - \frac{5\,199\,361\,041\,200\,909}{379\,725\,696\,000}\lambda^9 + \frac{47\,137\,018\,631\,639\,513}{1\,139\,177\,088\,000}\lambda^8 \\ & + \frac{1\,145\,930\,623\,773\,433}{341\,753\,126\,400}\lambda^{10} - \frac{3631}{605\,430}\lambda^{15} - \frac{21\,985\,862\,341}{3\,645\,600}p\lambda^5 + \frac{460\,650\,668\,593}{43\,747\,200}p\lambda^4 \\ & - \frac{154\,784\,798\,249}{13\,124\,160}p\lambda^3 + \frac{255\,035\,489\,398}{4\,944\,345}\lambda^2 + \frac{3\,925\,747\,081}{1\,749\,888}p\lambda^6 - \frac{90\,876\,950\,917}{15\,256\,836\,000}\lambda^{13} \\ & - \frac{1\,159\,012\,171}{2\,187\,360}p\lambda^7 - \frac{1\,743\,623\,839\,315\,721}{14\,239\,713\,600}\lambda^3 + \frac{30\,878}{315}p^2 + \frac{222\,361}{2520}p^2\lambda^4 \\ & - \frac{433\,453}{1008}p^2\lambda^3 + \frac{819\,967}{1008}p^2\lambda^2 - \frac{718\,133}{1260}p^2\lambda - \frac{3\,824\,219\,437}{1\,367\,100}p\lambda - \frac{3631}{630}p^2\lambda^5 \\ & + \frac{160\,864\,251\,357\,763\,979}{854\,382\,816\,000}\lambda^4 - \frac{101\,487\,682\,282\,697}{170\,876\,563\,200}\lambda^{11} - \frac{3\,412\,403}{585\,900}p\lambda^9 + \frac{3631}{19\,530}p\lambda^{10} \\ & + \frac{2\,474\,356\,747}{32\,810\,400}p\lambda^8 + \frac{4\,478\,040\,783\,667}{61\,027\,344\,000}\lambda^{12}, \end{aligned}$$

$$\begin{aligned} g_{51}(\lambda) = {} & -\frac{7\,991\,397\,801\,907\,001}{3\,218\,768\,595\,000}\lambda + \frac{430\,769\,061\,660\,938\,381}{51\,500\,297\,520\,000}\lambda^4 - \frac{90\,522\,930\,353\,255\,419}{794\,576\,018\,880\,000}\lambda^9 \\ & + \frac{7\,882\,042\,993\,003\,211}{397\,288\,009\,440\,000}\lambda^{10} + \frac{38\,819\,644\,979\,750\,780\,339}{11\,124\,064\,264\,320\,000}\lambda^6 \\ & + \frac{5\,294\,886\,380\,912\,311\,157}{11\,124\,064\,264\,320\,000}\lambda^8 - \frac{16\,491\,589\,988\,451\,048\,767}{11\,124\,064\,264\,320\,000}\lambda^7 \\ & - \frac{2\,550\,527\,148\,568\,185\,769}{309\,001\,785\,120\,000}\lambda^3 - \frac{68\,686\,980\,782\,797}{28\,377\,714\,960\,000}\lambda^{11} \\ & + \frac{440\,289\,519\,864\,500\,737}{77\,250\,446\,280\,000}\lambda^2 - \frac{7\,699\,195\,015\,471\,454\,567}{1\,236\,007\,140\,480\,000}\lambda^5 + \frac{3631}{18\,768\,330}\lambda^{14} \\ & - \frac{384\,159}{41\,707\,400}\lambda^{13} + \frac{30\,684\,351\,847}{157\,653\,972\,000}\lambda^{12} + \frac{23\,072\,498\,192\,986}{44\,705\,119\,375}, \end{aligned}$$

$$\tilde{g}_{12} = 0, \quad \tilde{g}_{22} = 2,$$

$$g_{42}(p,\lambda) = -\frac{22\,963\,886}{1\,177\,225} - \frac{6049}{1085}p - p^2 - \frac{237\,550\,583}{1\,367\,100}\lambda - \frac{113\,747}{1260}p\lambda + \frac{1\,277\,029\,067}{607\,600}\lambda^2$$
$$+ \frac{1\,419\,991}{2520}p\lambda^2 - \frac{1\,644\,438\,853}{234\,360}\lambda^3 - \frac{817\,177}{1008}p\lambda^3 + \frac{92\,476\,221\,137}{8\,202\,600}\lambda^4 + \frac{2\,164\,661}{5040}p\lambda^4$$
$$- \frac{14\,100\,715\,427\,003}{1\,356\,163\,200}\lambda^5 - \frac{6\,890\,671}{78\,120}p\lambda^5 + \frac{131\,464\,618\,651}{21\,873\,600}\lambda^6 + \frac{3631}{630}p\lambda^6$$
$$- \frac{3\,920\,013\,073}{1\,749\,888}\lambda^7 + \frac{1\,930\,062\,779}{3\,645\,600}\lambda^8 - \frac{2\,473\,263\,067}{32\,810\,400}\lambda^9 + \frac{105\,765\,593}{18\,162\,900}\lambda^{10} - \frac{3631}{19\,530}\lambda^{11},$$

$$g_{52}(\lambda) = -\frac{36\,874\,722\,147}{5\,109\,156\,500} - \frac{2\,362\,315\,264\,557}{20\,436\,626\,000}\lambda + \frac{20\,884\,081\,349\,269}{61\,309\,878\,000}\lambda^2$$
$$- \frac{538\,059\,413\,076\,769}{1\,103\,577\,804\,000}\lambda^3 + \frac{1\,793\,665\,758\,154\,211}{4\,414\,311\,216\,000}\lambda^4$$
$$- \frac{1\,445\,125\,981\,988\,557}{6\,621\,466\,824\,000}\lambda^5 + \frac{1\,026\,288\,639\,816\,701}{13\,242\,933\,648\,000}\lambda^6$$
$$- \frac{8\,405\,817\,164\,119}{472\,961\,916\,000}\lambda^7 + \frac{1\,174\,407\,170\,347}{472\,961\,916\,000}\lambda^8 - \frac{106\,666\,081}{563\,049\,900}\lambda^9$$
$$+ \frac{3631}{605\,430}\lambda^{10},$$

$$g_{43}(\lambda) = -\frac{63}{4}\lambda^5 + \frac{651}{8}\lambda^4 - \frac{1395}{8}\lambda^3 + \frac{651}{4}\lambda^2 - 63\lambda + \lambda^6 + 8,$$

$$g_{53}(\lambda) = -\frac{1}{31}\lambda^5 + \frac{31}{60}\lambda^4 - \frac{155}{56}\lambda^3 + \frac{155}{24}\lambda^2 - \frac{31}{4}\lambda + \frac{3879}{1085}.$$

(The form of the other elements is obvious.)

In this case, $d_1(p) = (p-2)(p-1)p(p+1)(p+2)(p+3)$ (see (28)). By [29, Theorem 1.1], the first 4 components of system (30) are degenerate.

3. Now we write the matrices of the finite stabilization controller (34)–(36) for system (48):

$$R_{00}(p,\lambda) = \Big[-\frac{1}{3}\lambda^4 + \frac{1}{2}\lambda^3 + \frac{4}{3}\lambda^2 - \lambda\Big], \quad R_{01}(p,\lambda) = \Big[\lambda^3 - \frac{7}{2}\lambda^2 + \frac{7}{2}\lambda - 1\Big],$$

$$R_{02}(p,\lambda) = \Big[-\frac{1}{3}\lambda^4 - \frac{1}{6}\lambda^3 + \frac{7}{3}\lambda^2 + \frac{5}{3}\lambda - 2, -\frac{2}{3}\lambda^2 + \lambda - \frac{4}{3}, 0, 0, 0\Big],$$

$$R_{10}(p,\lambda) = \Big[-\frac{2}{9}\lambda^3 + \frac{5}{9}\lambda^2 + \frac{4}{3}\lambda\Big], \quad R_{11}(p,\lambda) = \Big[\frac{2}{3}\lambda^2 - 3\lambda + \frac{7}{3}\Big],$$

$$R_{12}(p,\lambda) = \Big[-\frac{2}{9}\lambda^3 + \frac{1}{9}\lambda^2 + \frac{22}{9}\lambda + \frac{8}{3}, -\frac{4}{9}\lambda + \frac{10}{9}, 0, 0, 0\Big],$$

$$R_{22}(p,\lambda) = Q(p,\lambda), \quad K(\lambda) = \mathrm{col}\Big[0, 0, \frac{1}{2}\lambda^2 - \frac{1}{2}\lambda, 0, 0\Big].$$

We compose the characteristic matrix $W_1(p,\lambda)$ of the closed loop system (48), (34)–(36) (see the proof of Proposition 1). Direct verification shows that $|W_1(p,\lambda)| = d_1(p)d_0(p)$. By [29, Theorem 1.1], components nos. 1, 2, 4–7 of system (48), (34)–(36) become degenerate in time $16h$. (Here, 16 is the maximum degree of the variable λ of the polynomials representing the elements of the matrices of the controller (4).) Step 3) is completed. In this case, $\rho_0 = 2$; letting $\rho_1 = 2$ and $t_4 = 4h$ in Lemma 2, we observe that it is possible to take $t_6 = t_5 + t_4 + 4h = 10h$ since $m_0 = 2$ (see Lemma 2). Finally, the finite stabilization output-feedback controller is given by formula (38), and $t_1 = t_6 + 16h = 26h$ can be set in the identity (6).

4. CONCLUSIONS

In this paper, we have derived an existence criterion for a finite stabilization output-feedback controller as well as have proposed its design method. Conditions (7) and (8) represent [25, p. 206; 32] a complete 0-controllability criterion for system (1), (2) (a complete damping/calming criterion for this system). Conditions (9) and (10) are [25, p. 204; 32] represent a final observability criterion for system (1), (2) (i.e., the existence of a single-valued continuous operator for reconstructing the state of system (1) by the past output (2)). Thus, a finite stabilization output-feedback controller exists iff system (1), (2) is both completely 0-controllable and finally observable. The design procedure of a finite stabilization output-feedback controller is based on the methods for constructing controllers and observers [22, 25, 31], which involve algebraic operations implemented in most modern computer mathematics systems. Therefore, it is possible to automate the computational procedures proposed above when developing automatic control systems.

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APPENDIX

Proof of Lemma 1. If for any initial function φ in (3) there exists a control input u (a programmed or feedback law) ensuring (6), then system (1) is completely 0-controllable. Hence [32], conditions (7) and (8) are necessary. Let us establish the necessity of condition (9). Supposing the existence of a finite stabilization output-feedback controller of the form (4), we assume on the contrary that condition (9) is violated for some $p_0 \in \mathbb{C}$. Choosing a vector $g_0 \in \mathbb{C}^n$ as the solution of the algebraic system $W(p_0, e^{-p_0 h})g_0 = 0$, $C(e^{-p_0 h})g_0 = 0$, we define the function $x_{p_0}(t) = \mathbf{Re}(g_0 e^{p_0 t})$, $t \geqslant -mh$, if it is nonzero and $x_{p_0}(t) = \mathbf{Im}(g_0 e^{p_0 t})$, $t \geqslant -mh$, otherwise.

The controller (4) ensures the identity (6) regardless of the initial conditions (3) and (5). We set $\varphi(t) = x_{p_0}(t)$, $t \in [-mh, 0]$, and $\tilde{\varphi}(t) = 0$, $t \in [t_0 - \tilde{h}, t_0]$, in (3) and (5), respectively. The characteristic matrix of system (1), (4) is given by

$$W_1(p,\lambda) = \begin{bmatrix} W(p,\lambda) - B(\lambda)U_{11}(p,\lambda)C(\lambda) & -B(\lambda)U_{12}(p,\lambda) \\ -U_{21}(p,\lambda)C(\lambda) & pI_{\tilde{n}} - U_{22}(p,\lambda) \end{bmatrix}, \tag{A.1}$$

where $e^{-ph} = \lambda$. From (A.1) it follows that the function $\mathrm{col}[x_{p_0}(t), 0]$, $t > t_0$, is a nonzero solution of the closed loop system (1), (4). This obviously contradicts (6).

Now we show the necessity of condition (10). By the definition of a finite stabilization output-feedback controller, the spectrum of the system is finite, $|W_1(p,\lambda)| = w(p)$, where $w(p)$ is a polynomial. Consider the auxiliary system

$$(I_n - \Phi_0(\lambda_h))\dot{\xi}(t) = \Phi(\lambda_h)\xi(t) + \Psi(\lambda_h)\bar{u}(t), \quad t > 0, \tag{A.2}$$

where $\Phi_0(\lambda) = \big(D(\lambda)\big)'$, $\Phi(\lambda) = \big(A(\lambda)\big)'$, $\Psi(\lambda) = \big(C(\lambda)\big)'$, and $\bar{u}$ is a piecewise continuous control input. The initial conditions for system (A.2) are chosen similarly to those of (3).

For system (A.2) we define the controller

$$\begin{aligned} \bar{u}(t) &= H_{11}(p_D, \lambda_h)\xi(t) + H_{12}(p_D, \lambda_h)\tilde{x}(t), \\ \dot{\tilde{x}}(t) &= H_{21}(p_D, \lambda_h)\xi(t) + H_{22}(p_D, \lambda_h)\tilde{x}(t), \end{aligned} \tag{A.3}$$

where $H_{i1}(p,\lambda) = \big(B(\lambda)U_{1i}(p,\lambda)\big)'$ and $H_{i2}(p,\lambda) = \big(U_{2i}(p,\lambda)\big)'$, $i = 1, 2$. Let $W_\xi(p,\lambda)$ denote the characteristic matrix of system (A.2), (A.3). It is easy to see that $W_\xi(p,\lambda) = \big(W_1(p,\lambda)\big)'$, so

$|W_\xi(p, e^{-ph})| = w(p)$. Thus, there exists a feedback law for system (A.2) such that the closed loop system has a finite (but not a priori given) spectrum, i.e., it is spectrally reducible. Therefore [15], the condition $\operatorname{rank}[I_n - \Phi_0(\lambda),\ \Psi(\lambda)] = n\,\forall\lambda \in \mathbb{C}$ holds, which is equivalent to (10). The proof of Lemma 1 is complete.

Proof of Statement 1. The characteristic matrix $W_1(p, e^{-ph})$ of system (1), (2), (34)–(36) is given by

$$W_1(p,\lambda) = \begin{bmatrix} W(p,\lambda) - B(\lambda)R_{00}(p,\lambda)C(\lambda) & -B(\lambda)R_{01}(p,\lambda) & -B(\lambda)R_{02}(p,\lambda) \\ -R_{10}(p,\lambda)C(\lambda) & pI_{n_1} - R_{11}(p,\lambda) & -R_{12}(p,\lambda) \\ -(K(\lambda)+\overline{B}(\lambda)R_{00}(p,\lambda))C(\lambda) & -\overline{B}(\lambda)R_{01}(p,\lambda) & pI_{n_2} - R_{22}(p,\lambda) - \overline{B}(\lambda)R_{02}(p,\lambda) \end{bmatrix}, \tag{A.4}$$

where $\lambda = e^{-ph}$.

We represent the variable x_2 in the relations (34)–(36) as a vector with two vector components: $x_2 = \operatorname{col}[x_{21}, x_{22}]$, where $x_{21} \in \mathbb{R}^n$ and $x_{22} \in \mathbb{R}^3$. Also, we partition the matrices $R_{i2}(p,\lambda)$, $i = \overline{1,2}$, and $K(\lambda)$ in (37) into blocks corresponding to the components x_{21} and x_{22} and write them in an expanded form:

$$\begin{gathered} R_{02}(p,\lambda) = [L_{00}(p,\lambda)\Pi_{11}(\lambda), 0_{r\times 3}],\ R_{12}(p,\lambda) = [L_{10}(p,\lambda)\Pi_{11}(\lambda), 0_{n_1\times 3}], \\ R_{22}(p,\lambda) = \begin{bmatrix} A(\lambda)\Pi_{11}(\lambda) + V_{i_0}(\lambda)\widetilde{C}(\lambda) & Q_{12}(p,\lambda) \\ Q_{21}(p,\lambda) & Q_{22}(p,\lambda) \end{bmatrix},\ K(\lambda) = \begin{bmatrix} K_0(\lambda) \\ -K_1(\lambda) \end{bmatrix}. \end{gathered} \tag{A.5}$$

Here, the blocks $Q_{12}(p,\lambda) \in \mathbb{R}^{n\times 3}[\lambda]$, $Q_{21}(p,\lambda) \in \mathbb{R}^{3\times n}[\lambda]$, and $Q_{22}(p,\lambda) \in \mathbb{R}^{3\times 3}[\lambda]$ correspond to the block partition of the matrix $Q(p,\lambda)$ in (29) (the first upper block of the matrix (29) is the matrix $A_V(\lambda)$ described by (22)), and $K_1(\lambda) = \operatorname{col}[1,0,0]\tilde{e}'_{i_0}C_y(\lambda)$ (27).

Remark 7. Below it will be necessary to write the matrices partitioned into blocks. To fit them on the page width, thus making the considerations more visual, we will occasionally omit arguments in the notation of matrix blocks. For example, entries like B and $L_{00}\Pi_{11}$ will indicate $B(\lambda)$ and $L_{00}(p,\lambda)\Pi_{11}(\lambda)$, respectively.

Using the block partition (A.5) and the definitions of the matrices $K_0(\lambda)$ (22) and $\overline{B}(\lambda)$ (26), we write the matrix (A.4) as

$$W_1(p,\lambda) = \begin{bmatrix} W + BL_{00}\Pi_{12}C & -BL_{01} & -BL_{00}\Pi_{11} & 0_{n\times 3} \\ L_{10}\Pi_{12}C & pI_{n_1} - L_{11} & -L_{10}\Pi_{11} & 0_{n_1\times 3} \\ BL_{00}\Pi_{12}C + (A\Pi_{12} + V_{i_0}C_y)C & -BL_{01} & pI_n - A\Pi_{11} - V_{i_0}\widetilde{C} - BL_{00}\Pi_{11} & -Q_{12} \\ K_1C & 0_{3\times n_1} & -Q_{21} & pI_3 - Q_{22} \end{bmatrix}. \tag{A.6}$$

In system (1), (2), (34)–(36), let us introduce a new variable ε as follows:

$$x_{21}(t) = (I_n - D(\lambda_h))x(t) + \varepsilon(t), \quad t \geqslant t_0. \tag{A.7}$$

The change of variables (A.7) can be defined by the formulas

$$\begin{bmatrix} x(t) \\ x_1(t), \\ x_{21}(t) \\ x_{22}(t) \end{bmatrix} = \Omega(\lambda_h)\begin{bmatrix} x(t) \\ x_1(t), \\ \varepsilon(t) \\ x_{22}(t) \end{bmatrix}, \quad \Omega(\lambda) = \begin{bmatrix} I_n & 0_{n\times n_1} & 0_{n\times n} & 0_{n\times 3} \\ 0_{n_1\times n} & I_{n_1} & 0_{n_1\times n} & 0_{n_1\times 3} \\ I_n - D(\lambda) & 0_{n\times n_1} & I_n & 0_{n\times 3} \\ 0_{3\times n} & 0_{3\times n_1} & 0_{3\times n} & I_3 \end{bmatrix}, \quad |\Omega(\lambda)| \equiv 1.$$

Due to these formulas, the matrix $W_1(p,\lambda)\Omega(\lambda)$ will be the characteristic matrix obtained after the system replacement, and $|W_1(p,\lambda)| = |W_1(p,\lambda)\Omega(\lambda)|$.

Further transformations of the matrix $W_1(p,\lambda)\Omega(\lambda)$ require some relations. Note preliminarily that the definition of the matrices $\Pi_{ij}(\lambda)$ implies

$$\Pi_{11}(\lambda)(I_n - D(\lambda)) - \Pi_{12}(\lambda)C(\lambda) = I_n. \tag{A.8}$$

Next, in the matrix (A.6), we add block no. (3,3) multiplied on the right by the matrix $\big(I_n - D(\lambda)\big)$ to block no. (3,1). Using formula (A.8), we have the following chain of equalities:

$$\begin{gathered}
B(\lambda)L_{00}(p,\lambda)\Pi_{12}(\lambda)C(\lambda) + (A(\lambda)\Pi_{12}(\lambda) + V_{i_0}(\lambda)C_y(\lambda))C(\lambda)\\
+\Big(pI_n - A(\lambda)\Pi_{11}(\lambda) - V_{i_0}(\lambda)\widetilde{C}(\lambda) - B(\lambda)L_{00}(p,\lambda)\Pi_{11}(\lambda)\Big)(I_n - D(\lambda))\\
= B(\lambda)L_{00}(p,\lambda)\Pi_{12}(\lambda)C(\lambda) + A(\lambda)\Pi_{12}(\lambda)C(\lambda)\\
+V_{i_0}(\lambda)\begin{bmatrix}(I_l + C(\lambda)\Pi_{12}(\lambda))C(\lambda)\\ (I_n - D(\lambda))\Pi_{12}(\lambda)C(\lambda)\end{bmatrix} + p(I_n - D(\lambda))\\
-A(\lambda)\Pi_{11}(\lambda)(I_n - D(\lambda)) - V_{i_0}(\lambda)\begin{bmatrix}C(\lambda)\Pi_{11}(\lambda)(I_n - D(\lambda))\\ \Big((I_n - D(\lambda))\Pi_{11}(\lambda) - I_n\Big)(I_n - D(\lambda))\end{bmatrix}\\
-B(\lambda)L_{00}(p,\lambda)\Pi_{11}(\lambda)(I_n - D(\lambda)) = -B(\lambda)L_{00}(p,\lambda) + p(I_n - D(\lambda)) - A(\lambda)\\
+V_{i_0}(\lambda)\begin{bmatrix}C(\lambda) + C(\lambda)\Big(\Pi_{12}(\lambda)C(\lambda) - \Pi_{11}(\lambda)(I_n - D(\lambda))\Big)\\ (I_n - D(\lambda))\Big(\Pi_{12}(\lambda)C(\lambda) - \Pi_{11}(\lambda)(I_n - D(\lambda))\Big) + (I_n - D(\lambda))\end{bmatrix}\\
= W(p,\lambda) - B(\lambda)L_{00}(p,\lambda).
\end{gathered}\tag{A.9}$$

Then, in the matrix (A.6), we add the first row of block no. (4,3) multiplied on the right by the matrix $(I_n - D(\lambda))$ to the first row of block no. (4,1). (Note that the remaining two lower rows of the above blocks are zero, which follows from (29) and the form of the matrix $K_1(\lambda)$.) Using the intermediate reasoning in the chain of equalities (A.9), we arrive at the relation

$$\begin{gathered}
[1,0,0]K_1(\lambda)C(\lambda) - [1,0,0]Q_{21}(p,\lambda)(I_n - D(\lambda))\\
= \tilde{e}'_{i_0}\left(\begin{bmatrix}(I_l + C(\lambda)\Pi_{12}(\lambda))C(\lambda)\\ (I_n - D(\lambda))\Pi_{12}(\lambda)C(\lambda)\end{bmatrix} - \begin{bmatrix}C(\lambda)\Pi_{11}(\lambda)(I_n - D(\lambda))\\ \Big((I_n - D(\lambda))\Pi_{11}(\lambda) - I_n\Big)(I_n - D(\lambda))\end{bmatrix}\right) = 0.
\end{gathered}\tag{A.10}$$

Due to formula (A.8) and the relations (A.9) and (A.10),

$$W_1(p,\lambda)\Omega(\lambda) = \begin{bmatrix}W - BL_{00} & -BL_{01} & -BL_{00}\Pi_{11} & 0_{n\times 3}\\ -L_{10} & pI_{n_1} - L_{11} & -L_{10}\Pi_{11} & 0_{n_1\times 3}\\ W - BL_{00} & -BL_{01} & pI_n - A\Pi_{11} - V_{i_0}\widetilde{C} - BL_{00}\Pi_{11} & -Q_{12}\\ 0_{3\times n} & 0_{3\times n_1} & -Q_{21} & pI_3 - Q_{22}\end{bmatrix}.$$

In the matrix $W_1(p,\lambda)\Omega(\lambda)$, we multiply the first row of blocks by (-1) and add it to the third row, replacing the third row of blocks with the result. Let Ω_1 denote the matrix of this transformation. Obviously, $|\Omega_1| = 1$ and

$$\begin{gathered}
\Omega_1 W_1(p,\lambda)\Omega(\lambda) = \begin{bmatrix}W - BL_{00} & -BL_{01} & -BL_{00}\Pi_{11} & 0_{n\times 3}\\ -L_{10} & pI_{n_1} - L_{11} & -L_{10}\Pi_{11} & 0_{n_1\times 3}\\ 0_{n\times n} & 0_{n\times n_1} & pI_n - A\Pi_{11} - V_{i_0}\widetilde{C} & -Q_{12}\\ 0_{3\times n} & 0_{3\times n_1} & -Q_{21} & pI_3 - Q_{22}\end{bmatrix}\\
= \begin{bmatrix}W_0(p,\lambda) & \widetilde{W}(p,\lambda)\\ 0_{(n+1)\times(n+n_1)} & pI_{n+3} - Q(p,\lambda)\end{bmatrix},
\end{gathered}\tag{A.11}$$

where the block $\widetilde{W}(p,\lambda)$ is defined straightforwardly. The structure of the matrix (A.11) shows that the function $\mathrm{col}[\varepsilon, x_{22}]$ is defined by a system with the characteristic matrix $I_{n+3} - Q(p,\lambda)$ (i.e., by system (30), which is pointwise degenerate). Therefore, $e_i'\mathrm{col}[\varepsilon(t), x_{22}(t)] \equiv 0$, $t \geqslant t_0 + \tilde{t}_2$, $i = \overline{1, n+2}$. So, for $t \geqslant \bar{t}_4$, we have $\bar{t}_4 = t_0 + \tilde{t}_2 + \gamma_5 h$, where γ_5 is the maximum degree of the variable λ in the block $\widetilde{W}(p,\lambda)$, and the function $\mathrm{col}[x, x_1]$ is defined by a homogeneous system with the characteristic matrix (14), which is also pointwise degenerate. Hence, for $t_1 = \bar{t}_1 + \bar{t}_4$, where $\bar{t}_1$ is given by (12), the identities $\bar{e}_i'\mathrm{col}[x(t), x_1(t)] \equiv 0$, $t \geqslant t_1$, hold. In combination with (A.7), this result implies the pointwise degeneracy of system (1), (2), (34)–(36).

Due to the form of the matrix $\Omega_1 W_1(p,\lambda)\Omega(\lambda)$ in (A.11) and equalities (28) and (13), the eigenvalues of system (1), (2), (34)–(36) are determined by the roots of the polynomial $d_1(\lambda)d_0(\lambda)$. The proof of Proposition 1 is complete.

Proof of Lemma 2. Let us introduce the new variable $X(t) = \big(I_n - D(\lambda_h)\big)x(t)$, $t \geqslant 0$, in system (1). Then $x(t) = \Pi_D(\lambda_h)X(t)$, $t \geqslant h\deg_\lambda \Pi_D(\lambda)$, and the function $X(t)$ is defined by the delayed system

$$\dot{X}(t) = A(\lambda_h)\Pi_D(\lambda_h)X(t), \quad t > hm_0. \tag{A.12}$$

As is known, the smoothness of the solution of the delayed system (A.12) increases by one when increasing the time variable by the value $m_0 h$. Therefore, for the given number ρ_1 and $t \geqslant m_0 h + (\rho_1 - 1)m_0 h = \rho_1 m_0 h$, the function $X(t)$ is such that $X \in \widetilde{C}^{\rho_1}([\rho_1 m_0 h, +\infty), \mathbb{R}^n)$, and the desired conclusion follows. The proof of this lemma is complete.

Proof of Lemma 3. By condition (10), there exist [15; 25, p. 228] polynomial matrices $M_{ij}(\lambda)$ and $K_{ij}(\lambda)$ of appropriate dimensions such that

$$\begin{gathered}\begin{vmatrix} I_n - D(\lambda) - \lambda B(\lambda)M_{11}(\lambda) & -\lambda B(\lambda)M_{12}(\lambda) \\ -\lambda M_{21}(\lambda) & I_r - \lambda M_{22}(\lambda)\end{vmatrix} \equiv 1, \\ \begin{vmatrix} I_n - D(\lambda) - \lambda K_{11}(\lambda)C(\lambda) & -\lambda K_{12}(\lambda) \\ -\lambda K_{21}(\lambda)C(\lambda) & I_l - \lambda K_{22}(\lambda)\end{vmatrix} \equiv 1.\end{gathered} \tag{A.13}$$

We define the matrices

$$\widetilde{U}_{11}(\lambda) = 0_{r\times n}, \quad \widetilde{U}_{12}(\lambda) = \Big[\lambda M_{12}(\lambda),\ \ \lambda M_{11}(\lambda),\ \ 0_{n\times l}\Big],$$

$$\widetilde{U}_{21}(\lambda) = \begin{bmatrix} 0_{(r\times n)} \\ -\lambda K_{11}(\lambda) \\ -\lambda K_{21}(\lambda)\end{bmatrix},$$

$$\widetilde{U}_{22}(\lambda) = \begin{bmatrix} \lambda M_{22}(\lambda) & \lambda M_{21}(\lambda) & 0_{r\times l} \\ \lambda B(\lambda)M_{12}(\lambda) & D(\lambda) + \lambda K_{11}(\lambda)C(\lambda) + \lambda B(\lambda)M_{11}(\lambda) & \lambda K_{12}(\lambda) \\ 0_{l\times r} & \lambda K_{21}(\lambda)C(\lambda) & \lambda K_{22}(\lambda)\end{bmatrix}.$$

Note that $\widetilde{U}_{ij}(0)$ are zero matrices. Let us denote

$$\Gamma(\lambda) = E\big(I_{2n+r+l} - \widetilde{D}(\lambda)\big)E^{-1}, \ \text{ where } \ E = \begin{bmatrix} I_n & 0_{n\times r} & 0_{n\times n} & 0_{n\times l} \\ 0_{r\times n} & I_r & 0_{r\times n} & 0_{r\times l} \\ -I_n & 0_{n\times r} & I_n & 0_{n\times l} \\ 0_{l\times n} & 0_{l\times r} & 0_{l\times n} & I_l\end{bmatrix}.$$

Direct verification shows that

$$\Gamma(\lambda)=\begin{bmatrix} I_n-D(\lambda)-\lambda B(\lambda)M_{11}(\lambda) & -\lambda B(\lambda)M_{12}(\lambda) & -\lambda B(\lambda)M_{11}(\lambda) & 0_{n\times l}\\ -\lambda M_{21}(\lambda) & I_r-\lambda M_{22}(\lambda) & -\lambda M_{21}(\lambda) & 0_{r\times l}\\ 0_{n\times n} & 0_{n\times r} & I_n-D(\lambda)-\lambda K_{11}(\lambda)C(\lambda) & -\lambda K_{12}(\lambda)\\ 0_{l\times n} & 0_{l\times r} & -\lambda K_{21}(\lambda)C(\lambda) & I_l-\lambda K_{22}(\lambda)\end{bmatrix}.$$

In view of the identities (A.13), we conclude that $|\Gamma(\lambda)|\equiv 1$, and the relation (42) is immediate. The proof of this lemma is complete.

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